An automatic control point choice in algebraic numerical grid generation

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Abstract

A strategy to construct a grid conforming to the boundaries of a prescribed domain by using transfinite interpolation methods is discussed. A transfinite interpolation procedure is combined with a B-spline tensor product scheme defined by using suitable control points. Their choice is performed by taking into account a quality measure parameter based on the condition number of matrices linked to the covariant metric tensors.

1 Introduction

The algebraic grid generation approach relies on the construction of a coordinate transformation from the computational domain into the physical domain. In particular, this can be obtained through transfinite interpolating operators allowing us the generation of grids with boundary conformity. Furthermore, using a Hermite-type transfinite interpolating scheme we can obtain orthogonal grid lines emanating from the boundary. This can be very important for practical reasons since the grid point distribution in the immediate neighborhood of the boundaries has a strong influence on the accuracy of the numerical solution of partial differential equations [5]. Furthermore, in case a domain decomposition is necessary the orthogonality guarantees smoother grids. In order to obtain a grid with other specified properties, e.g. the control of the shape and position of the coordinate curves, transfinite interpolating methods can be combined with tensor product schemes using suitably chosen control points (see for instance [1, 2, 6, 7, 8]). Even though this type of algebraic method is computationally efficient, to define workable meshes, a significant amount of user interaction is required for the selection of the control points involved in the tensor product. To overcome this drawback, an automatic strategy for choosing the control points turns out to be desirable. Here, following the approach first discussed in [1], we present an algebraic Hermite-type transfinite method to construct a grid interpolating the boundary and its normal derivatives. In fact, given a "quadrilateral" domain $\Omega \subset \mathbb{R}^2$, a transformation $G: \; R = [0,1] \times [0,1] \rightarrow \Omega$ is defined as

$$G(s,t) := T_P(s,t) + (P_1 \oplus P_2)([\phi, \psi] - T_P)(s,t) \tag{1.1}$$

where T_P is a tensor product surface i.e. $T_P(s,t) := \sum_{i=1}^{m} \sum_{j=1}^{n} Q_{ij} B_{i,3}(s) B_{j,3}(t)$ with $B_{i,3}$ denoting the usual cubic B-spline, ϕ and ψ are boundary curves and $(P_1 \oplus P_2)$ is the

Boolean sum of Hermite-type blending function linear operators. The set $\mathcal{Q} = \{Q_{ij},\ i = 1,\dots,m,\ j = 1,\dots,n\}$ is the set of control points.

As already noted, the choice of the control points is a crucial matter. In this paper we take into account a grid quality measure parameter for their selection. In particular, the proposed automatic procedure relies on the fact that some grid properties can be described in terms of the condition number of matrices linked to the covariant metric tensors [4]. Therefore, the control points are chosen minimizing their condition number.

The outline of this paper is as follows. In Section 2, the transformation (1.1) is given in detail and its properties are investigated. In Section 3, a way for choosing the control points is proposed relying on a particular quality measure parameter. Finally, in Section 4 some numerical results are presented to illustrate the features of the proposed strategy.

2 The transformation

In this section the transformation (1.1) is characterized. Let us consider a "quadrilateral" domain $\Omega \subset \mathbb{R}^2$ such that $\partial\Omega = \cup_{i=1}^4 \partial\Omega_i$, with $\partial\Omega_1, \partial\Omega_2, \partial\Omega_3, \partial\Omega_4$ being the supports of four regular curves $\gamma_i : [0,1] \to \partial\Omega_i,\ i = 1,\dots,4$ taken counterclockwise. Furthermore, let us suppose that $\partial\Omega_1 \cap \partial\Omega_3 = \emptyset$ and $\partial\Omega_2 \cap \partial\Omega_4 = \emptyset$, with any other intersection occuring only at the end points of the boundary curves. In particular, the following compatibility conditions are assumed

$$\gamma_1(0) = \gamma_4(1),\ \gamma_1(1) = \gamma_2(0),\ \gamma_2(1) = \gamma_3(0),\ \gamma_4(0) = \gamma_3(1)\ .$$

For later convenience, we set $\phi_1(s) := \gamma_1(s),\ \phi_2(s) := \gamma_3(1-s)$ denoting by s the curve parameter running on $[0,1]$ and we set $\psi_1(t) := \gamma_4(1-t),\ \psi_2(t) := \gamma_2(t)$ denoting by t the curve parameter running on $[0,1]$. In addition, the components of the $\boldsymbol{\phi}$-curves and $\boldsymbol{\psi}$-curves are denoted by $\boldsymbol{\phi}^x, \boldsymbol{\phi}^y$ and $\boldsymbol{\psi}^x, \boldsymbol{\psi}^y$ respectively.

Next, we define four additional curves by computing the derivatives of the $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$-curves, i.e.,

$$\begin{aligned} \phi_{i+2}(s) &= \frac{\mathcal{C}}{\|\phi_i'\|_2}(-(\phi_i^y(s))',\ (\phi_i^x(s))')\ , \quad i = 1,2\ , \\ \psi_{j+2}(t) &= \frac{\mathcal{C}}{\|\psi_j'\|_2}(-(\psi_j^y(t))',\ (\psi_j^x(t))')\ , \quad j = 1,2\ , \end{aligned} \tag{2.1}$$

with $\mathcal{C}$ a constant value also depending on the curve orientations and with $\|\cdot\|_2$ the Euclidean norm. Then, we introduce the linear operators

$$\begin{aligned} P_1[\boldsymbol{\phi}](s,t) &:= \textstyle\sum_{i=1}^4 \alpha_i(t)\phi_i(s)\ , \quad P_2[\boldsymbol{\psi}](s,t) := \sum_{j=1}^4 \alpha_j(s)\psi_j(t)\ , \\ P_1P_2[\boldsymbol{\phi},\boldsymbol{\psi}](s,t) &:= \textstyle\sum_{i=1}^2 \left(\alpha_i(t)P_2[\boldsymbol{\psi}](s,u_i) + \alpha_{i+2}(t)\frac{\partial P_2[\boldsymbol{\psi}](s,u_i)}{\partial t}\right)\ , \end{aligned} \tag{2.2}$$

where $u_1 = 0,\ u_2 = 1$. The functions $\alpha_i,\ i = 1,\dots,4$, are the dilated versions of the classical Hermite bases with support on $[0,\bar{u}]$ and on $[1-\bar{u},1]$ being $0<\bar{u}<1$, i.e.

$$\alpha_1(s) := (1+2\tfrac{s}{\bar{u}})(1-\tfrac{s}{\bar{u}})^2 , \qquad \alpha_3(s) := s(1-\tfrac{s}{\bar{u}})^2 , \quad s\in[0,\bar{u}],$$

$$\alpha_2(s) := (3-2\tfrac{s+\bar{u}-1}{\bar{u}})(\tfrac{s+\bar{u}-1}{\bar{u}})^2 , \quad \alpha_4(s) := (s-1)(\tfrac{s+\bar{u}-1}{\bar{u}})^2 , \quad s\in[1-\bar{u},1]. \tag{2.3}$$

The Boolean sum operator $(P_1\oplus P_2) = P_1+P_2-P_1P_2$ provides the blending function surface

$$B(s,t) := (P_1\oplus P_2)[\phi,\psi](s,t) = P_1[\phi](s,t)+P_2[\psi](s,t)-P_1P_2[\phi,\psi](s,t) . \tag{2.4}$$

It is known that B satisfies

$$\begin{array}{ll} B(u_i,t)=\psi_i(t) \ , \ i=1,2 & \frac{\partial B(u_i,t)}{\partial s}=\psi_i(t) \ , \ i=3,4 \ , \\ B(s,w_j)=\phi_j(s) \ , \ j=1,2 & \frac{\partial B(s,w_j)}{\partial t}=\phi_j(s) \ , \ j=3,4 \ , \end{array} \tag{2.5}$$

where $u_1=u_3=0$, $u_2=u_4=1$ and $w_1=w_3=0$, $w_2=w_4=1$. It is worthwhile to remark that, as we are dealing with orthogonal grid lines emanating from the boundary of the domain, the intersecting boundary curves must be also orthogonal. Thus, the following additional conditions are assumed:

$$\begin{array}{llll} \phi_{i+2}(0) & =\psi_1^{'}(w_i), & \phi_{i+2}(1) & =\psi_2^{'}(w_i) \ , \\ \psi_{i+2}(0) & =\phi_1^{'}(u_i), & \psi_{i+2}(1) & =\phi_2^{'}(u_i) \ , \\ \phi_i^{''}(0) & =\psi_1^{''}(w_i), & \phi_i^{''}(1) & =\psi_2^{''}(w_i) \ , \end{array} \qquad i=1,2 \ . \tag{2.6}$$

Now, in order to define a suitable grid, following the approach given in [1], we use the linear transformation G

$$G(s,t) := T_P(s,t)+(P_1\oplus P_2)\big([\phi,\psi]-T_P\big)(s,t) \tag{2.7}$$

where $T_P(s,t) := \sum_{i=1}^{m}\sum_{j=1}^{n} Q_{ij}B_{i,3}(s)B_{j,3}(t)$ with $B_{i,3}$ denoting the usual cubic B-splines with uniform knots. The set $\mathcal{Q}=\{Q_{ij},\ i=1,\dots,m,\ j=1,\dots,n\}$ is a suitable set of control points whose definition is discussed in Section 3. It should be noted that in (2.7) the Boolean sum operator is also acting on a surface $T_P(s,t)$. In this case (2.2) is used taking the eight boundary curves $T_P(0,t)$, $T_P(1,t)$, $T_P(s,0)$, $T_P(s,1)$, $\frac{\partial T_P(0,t)}{\partial s}$, $\frac{\partial T_P(1,t)}{\partial s}$, $\frac{\partial T_P(s,0)}{\partial t}$, $\frac{\partial T_P(s,1)}{\partial t}$.

It is easy to show that G still satisfies $G(u_i,t)=\psi_i(t)$, $i=1,2$, $\frac{\partial G(u_i,t)}{\partial s}=\psi_i(t)$, $i=3,4$, $G(s,w_j)=\phi_j(s)$, $j=1,2$ and $\frac{\partial G(s,w_j)}{\partial t}=\phi_j(s)$, $j=3,4$. Furthermore, because of the locality of the blending functions α_i, $i=1,\dots,4$, the control of the coordinate lines obtained by means of the evaluation of G over a parameter set in the interior of the domain is mainly based on the contribution of T_P. This fact and the use of B-splines ensures the convex-hull property in the interior of the domain. This property is of importance in numerical grid generation to locate the grid with respect to the position of control points.

3 Grid quality measure

It is well known that grid generation techniques sensible to grid quality features are particularly attractive. Thus, in this section, we discuss a strategy to choose the set $\mathcal{Q}$ of control points based on a suitable grid quality measure parameter.

Given a set of grid points $\mathcal{G} := \{G_{ij}\}_{i,j=1}^{M,N}$ defining the quadrilateral cells $\{C_{ij}\}_{i,j=1}^{M-1,N-1}$, quality measures can commonly include: grid "skewness", measuring the departure of C_{ij} from a rectangle, grid "aspect ratio", measuring the departure of C_{ij} from a rhombus or grid "conformality", measuring the departure of C_{ij} from a square (see for instance [5]).

Here, as done in [4] for the case of unstructured grids, we define a grid quality measure taking into account the condition number of particular matrices derived from the grid. As explained below, somehow this quality parameter measures the departure of C_{ij} from a square.

The strategy starts with a set $\mathcal{Q}^i$ of control points obtained by evaluating on a coarse parameter set $\mathcal{S}_c = \{(s_i, t_j)\}_{i,j=1}^{M_c,N_c}$ a Lagrange blending function surface (for detail related to Lagrange blending function methods we refer, for instance, to [3]) by working only with the four boundary curves of the given domain. Then, using $\mathcal{Q}^i$ a first grid is obtained by evaluating the surface G in (2.7) on a fine parameter set $\mathcal{S}_f = \{(s_i, t_j)\}_{i,j=1}^{M,N}$ obtaining the grid points

$$\mathcal{G} := \{G_{i,j} = (G^x_{i,j}, G^y_{i,j}) = G(s_i, t_j),\ i = 1, \ldots, M,\ j = 1, \ldots, N\}\ .$$

The set $\mathcal{G}$ is then used to define $(M-1) \times (N-1)$ bidimensional matrices associated with the $(M-1) \times (N-1)$ quadrilateral cells $C_{ij},\ i = 1, \ldots, M-1,\ j = 1, \ldots, N-1$. These matrices are defined as

$$A_{i,j} := \begin{pmatrix} G^x_{i+1,j} - G^x_{i,j} & G^x_{i,j+1} - G^x_{i,j} \\ G^y_{i+1,j} - G^y_{i,j} & G^y_{i,j+1} - G^y_{i,j} \end{pmatrix},\ i = 1, \ldots, M-1,\ j = 1, \ldots, N-1 \quad (3.1)$$

and their condition number $K(A_{i,j})$ is related to the stretch of the cells. In fact, it is easy to prove that $K(A_{i,j}) := ||A_{i,j}||_2 \cdot ||A^{-1}_{i,j}||_2 = 1$ if and only if we are dealing with a cell C_{ij} where the three points $G_{i,j+1}, G_{i,j}, G_{i+1,j}$ generate half a square [9]. On the other hand, in order to involve all the grid points in the quality measure it is also convenient to define the boundary matrices

$$\begin{aligned}
A^l_{i,N-1} &:= \begin{pmatrix} G^x_{i+1,N} - G^x_{i,N} & G^x_{i,N-1} - G^x_{i,N} \\ G^y_{i+1,N} - G^y_{i,N} & G^y_{i,N-1} - G^y_{i,N} \end{pmatrix},\ i = 1, \ldots, M-1, \\
A^r_{M-1,j} &:= \begin{pmatrix} G^x_{M,j+1} - G^x_{M,j} & G^x_{M-1,j} - G^x_{M,j} \\ G^y_{M,j+1} - G^y_{M,j} & G^y_{M-1,j} - G^y_{M,j} \end{pmatrix},\ j = 1, \ldots, N-1, \qquad (3.2) \\
A^u_{M-1,N-1} &:= \begin{pmatrix} G^x_{M,N} - G^x_{M-1,N} & G^x_{M,N} - G^x_{M,N-1} \\ G^y_{M,N} - G^y_{M-1,N} & G^y_{M,N} - G^y_{M,N-1} \end{pmatrix},
\end{aligned}$$

so that the boundary points are also taken into account.

Next, we modify the initial set $\mathcal{Q}^i$ of control points minimizing the following objective function

$$f_{ob} = \frac{1}{MN}\left(\sum_{i=1}^{M-1}\sum_{j=1}^{N-1} K(A_{i,j}) + \sum_{i=1}^{M-1} K(A^l_{i,N-1}) + \sum_{j=1}^{N-1} K(A^r_{M-1,j}) + K(A^u_{M-1,N-1})\right). \tag{3.3}$$

The minimization is done with respect to the control points under suitable constraints on their coordinates depending on the geometry of the domain Ω. This is the only user interaction required.

Obviously, since ideal inner cells are characterized by an associated matrix $A_{i,j}$ having a condition number close to one, the optimal distribution of the control points should guarantee $\min_{\mathcal{Q}} f_{ob} \approx 1$. On the other hand, $\min_{\mathcal{Q}} f_{ob}$ strongly depends on the geometry of the domain (for example in case of a squared domain the optimal value is $\min_{\mathcal{Q}} f_{ob} = 1$ while, in general, this value is not reached).

Summary of the Method

(1) Compute the initial set of control points $\mathcal{Q}^i$ by means of a Lagrange blending function method using the four given boundary curves,

(2) Compute the initial grid $\mathcal{G}^i = \{G(s_i, t_j),\ i = 1, \ldots, M,\ j = 1, \ldots, N\}$ with G given in (2.7) by using the set of control points $\mathcal{Q}^i$,

(3) Minimize the objective function (3.3) so defining a new set of control points $\mathcal{Q}^f$,

(4) Compute the final grid $\mathcal{G}^f = \{G(\tilde{s}_i, \tilde{t}_j),\ i = 1, \ldots, \tilde{M},\ j = 1, \ldots, \tilde{N}\}$ with G given in (2.7) by using the set of control points $\mathcal{Q}^f$ with $\tilde{M} \gg M$, $\tilde{N} \gg N$.

Remark 3.1 *We note that, in order to reduce the computational cost of the minimization procedure, the integers M and N are chosen less than $\tilde{M}$ and $\tilde{N}$.*

4 Numerical Results

We conclude the paper giving some numerical results testing the properties of the transformation G and showing the performance of the proposed approach.

Three domains are considered. For each of them we present the initial grid obtained by the transformation G using the initial set of control points $\mathcal{Q}^i$ and the final grid obtained using the set of control points $\mathcal{Q}^f$ resulting from the minimization procedure. In all the figures the control points are denoted by the symbol '*'. The minimization problem is solved by using a sequential quadratic programming method i.e. by using the routine constr of the Optimization toolbox of the Matlab package. In the minimization procedure, the constraints on the control points $\mathcal{Q}^f$ are chosen so that some geometric properties of the domain, such as symmetry and convexity, are preserved. Furthermore, in all the examples M and N are equal to $\tilde{M}$ and to $\tilde{N}$. The values of the objective function before the minimization (f^i_{ob}) and after the minimization (f^f_{ob}) are also given in the figure captions.

The first and the second test display a "waterway" grid and a $\mathcal{H}$-shaped grid with their control points before and after the minimization procedure. The effectiveness of the method is evident.

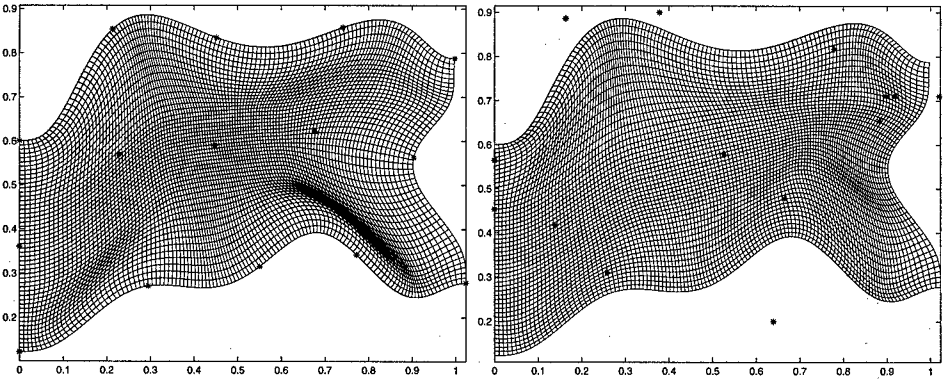

FIG. 1. Initial grid (left) and final grid (right), $f_{ob}^{i} = 3.74$, $f_{ob}^{f} = 1.65$.

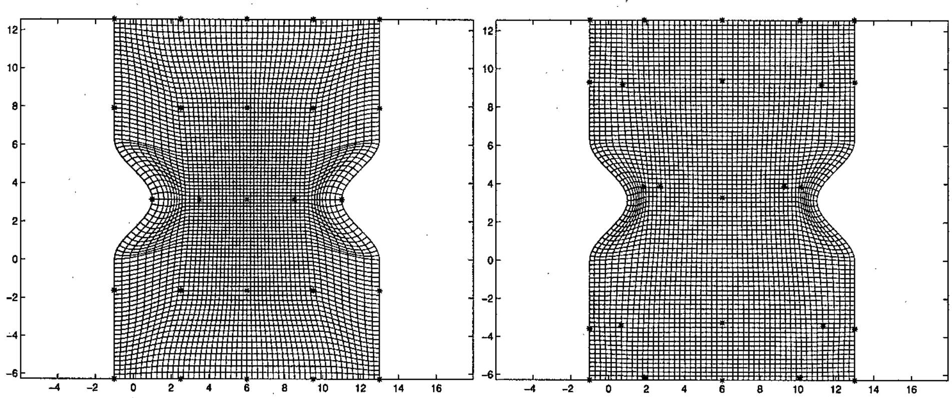

FIG. 2. Initial grid (left) and final grid (right), $f_{ob}^{i} = 1.45$, $f_{ob}^{f} = 1.22$.

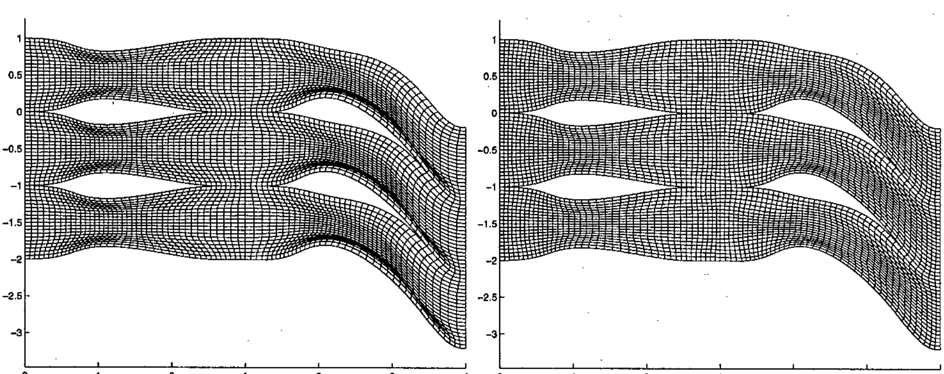

FIG. 3. Initial grid (left) and final grid (right), $f_{ob}^{i} = 1.89/6.36$, $f_{ob}^{f} = 1.59/2.20$.

Figure 3 shows a grid composed of six sub-grids, obtained via a domain decomposition approach. In this case, the Hermite-type interpolation method guarantees a C^1 connection among the patches. Here, the two values of f^i_{ob} and f^f_{ob} in the figure captions refer to the "horizontal" and "slanted" grids, respectively.

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